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Accurate and efficient waveforms for compact binaries on eccentric orbits

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Compact binaries that emit gravitational waves in the sensitivity band of ground-based detectors can have non-negligible eccentricities just prior to merger, depending on the formation scenario. We develop a purely analytic, frequency-domain model for gravitational waves emitted by compact binaries on orbits with small eccentricity, which reduces to the quasi-circular post-Newtonian approximant **TaylorF2** at zero eccentricity and to the post-circular approximation of Yunes et al. (2009) at small eccentricity. Our model uses a spectral approximation to the (post-Newtonian) Kepler problem to model the orbital phase as a function of frequency, accounting for eccentricity effects up to $\mathcal{O}(e^8)$ at each post-Newtonian order. Our approach accurately reproduces an alternative time-domain eccentric waveform model for $e \in [0, 0.4]$ and binaries with total mass $\lesssim 12M_\odot$. As an application, we evaluate the signal amplitude that eccentric binaries produce in different networks of existing and forthcoming gravitational waves detectors. Assuming a population of eccentric systems containing black holes and neutron stars that is uniformly distributed in co-moving volume, we estimate that second generation detectors like Advanced LIGO could detect approximately 0.1–10 events per year out to redshift $z \sim 0.2$, while an array of Einstein Telescope detectors could detect hundreds of events per year to redshift $z \sim 2.3$.

I. INTRODUCTION

Numerical studies suggest that, depending on the metallicity and chemical composition, the end point of stellar evolution for most massive stars above $\gtrsim 7M_\odot$ will be either a stellar mass black hole (BH) or a neutron star (NS) [1]. Binary systems of stellar-mass compact objects are the most promising sources of gravitational waves (GWs) for second generation ground-based interferometric detectors, such as Advanced LIGO (aLIGO), Virgo, LIGO India, and Kagra [2–4], as well as for planned third generation detectors like the Einstein Telescope (ET) [5]. Most previous analyses have assumed that these compact binaries will be in nearly quasi-circular orbits by the time they reach the sensitive frequencies of these detectors (i.e., orbital frequencies greater than a few Hz). Such assumptions were made because GW emission rapidly reduces the eccentricity of a binary system [6], and thus most astrophysical binaries that were formed at large separations and low frequencies will circularize well before their signal enters the detector’s sensitive band.

Many recent investigations, however, now suggest that there may exist several plausible, observationally-unconstrained astrophysical mechanisms through which GWs emitted by compact binaries with significant eccentricity may persist into the detector’s sensitivity band [7–10]. Dense stellar environments in galactic nuclei can facilitate frequent interactions, enabling direct dynamical capture [10–16] into high-eccentricity orbits via single-

single, binary-binary and binary-single processes [8]. A high density of compact objects in the galactic center cusp is expected on theoretical grounds, from mass segregation of individual objects [17, 18] and satellite stellar systems (e.g., clusters and small galaxies) [18, 19]. The compact object density, and hence the event rate, remains highly uncertain, but [19, 20] suggest $\gtrsim 10^3$ BHs in the central 0.1 pc of our galaxy, consistent with the high event rates quoted in [11, 12].

Globular clusters also provide an environment for stars and binaries to interact and form eccentric binaries. While typical interactions occur at large separations, binary-single and binary-binary interactions can produce close encounters that form high-eccentricity systems even for relatively low stellar velocities [8]. Additionally, these environments naturally form hierarchical triples, whose secular interactions (the Kozai-Lidov mechanism [21, 22]) can drive the inner binary to high eccentricity [9, 14, 23–25]. This effect also naturally occurs in binaries bound to supermassive BHs, and could therefore yield eccentric systems in galactic centers as well [7]. While relatively few BHs are known to exist in globular clusters, recently several accreting BH candidates have been found in extragalactic globular clusters [26, 27], and more recently in the Milky Way globular cluster M22 [28]. Recent numerical calculations also lend support to the idea that a significant population of BHs will persist in clusters for the full cluster lifetime [29, 30].

Another potential mechanism for forming eccentric ultra-compact binaries is tidal capture. NSs can absorb orbital energy during close passages, i.e., capture via closest approach for pericenter distances $r_p \simeq \mathcal{O}(10 -$

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100 km)($M/10M_\odot$) [31]. The tidal capture event rate is very sensitive to assumptions made about the retention rate of NSs [32], the fraction of NSs in the core of the globular cluster [33], BH ejection [24, 34–37] and the tidal capture cross section used.

A. Previous Work

All of the above scenarios are quite uncertain; hence, even a null GW result will significantly constrain previously inaccessible astrophysics. A null result could also arise from selection biases against eccentric binaries – for example, due to systematic errors in the waveform modeling if, for example, one tried to extract eccentric signals with quasi-circular waveforms. Selection biases are estimated by characterizing the “effectualness” of model A for finding members of model B: a candidate signal from B is compared with a dense, discrete, complete sample of A. Using leading-order post-Newtonian (PN) expansions, i.e., an expansion in small velocities and weak fields, for eccentric binary systems, Ref. [38] concluded that quasi-circular templates could effectively detect low-eccentricity compact binary sources: nonspinning searches were “effectual” for eccentric binaries. Subsequent studies quantified the selection bias against eccentric binaries [39–41]. For example, in initial LIGO [39], targeting binary NSs with quasi-circular templates would lead to a detection loss $\gtrsim 10\%$ for binaries with eccentricities $e \gtrsim 0.05$ at a Keplerian mean orbital frequency of 20 Hz. By contrast, in this paper we perform “faithfulness” studies, demonstrating that eccentricity has a distinguishable impact by comparing GWs from otherwise identical binaries with eccentricity to those without.

The studies mentioned above motivated the development of accurate eccentric waveform models to target compact binaries on eccentric orbits. Several authors over the years have generated time-domain eccentric waveforms, including kludge waveforms typically applied at high mass-ratio [42–49], precessing time-domain signals [50–56], and finally relatively high-accuracy time-domain PN models [57]. Though useful for qualitative work, these calculations faced some limitations. First, PN approximations become inaccurate when the orbital velocity becomes large enough, which can occur during pericenter passages for certain orbits, but always occurs close to merger. In such instances, numerical relativity was required to build confidence in the associated dynamical evolution and waveform models [58–60]. Second, the construction of such models is computationally expensive because of the need to solve ordinary differential equations in the time-domain with a very fine and constant discretization, so that aliasing and Nyquist noise is under control when computing the discrete Fourier transform (DFT) of the GW response function.

These limitations have been recently addressed in two different waveform models: the *x-model* of [59] and the

post-circular (PC) model of [61]. The former is a time-domain model with conservative orbital dynamics accurate to 3 PN¹ order and radiation-reaction accurate to 2 PN order. The *x-model* has been validated against one numerical relativity simulation of an equal-mass BH-BH with initial eccentricity $e = 0.1$, 21 GW cycles before merger [59]. The *x-model*, however, is quite computationally expensive (mainly because of the need to solve for the orbital evolution in the time-domain and then to Fourier transform the resulting response function to the frequency domain) [41] and not sufficiently accurate to model low-mass binary inspirals [63].

The PC model is a frequency-domain approach, where the conservative and dissipative orbital dynamics are treated in the PN approximation, but further expanded in a small eccentricity approximation through an analytic, high-order spectral decomposition [61]. The frequency-domain response function is then computed through the stationary-phase approximation (SPA) [61, 64]. Although in principle this model can be implemented to arbitrary PN order, only the leading PN order terms were included explicitly in [61], while 1 PN corrections [65], 2 PN corrections [66] and 3 PN corrections [57, 67] are now available to extend [61].

B. Executive Summary

In this work we develop the enhanced post-circular (EPC) model. This model is an extension of the PC analysis in [61], designed to reproduce in the zero-eccentricity limit the **TaylorF2** model at 3.5 PN order and to reproduce in the small eccentricity limit the PC model to leading (Newtonian) order. The **TaylorF2** model is a waveform family constructed from the PN approximation for non-spinning, quasi-circular binaries directly in the frequency-domain, using the SPA. Furthermore, as shown in [68, 69], **TaylorF2** 3.5 PN is accurate and computationally efficient to construct effectual searches of quasi-circular binary systems with total mass $\lesssim 12M_\odot$ [68, 69]. It is worth emphasizing that we could also try to reproduce the evolution of the time domain PN-based approximant **TaylorT4** 3.5 PN in the quasi-circular limit, since this model provides an accurate representation of the evolution of comparable mass quasi-circular binaries [63]. Nonetheless, anticipating that the matched-filtering in future GW searches will be carried out in the frequency domain, we provide a waveform family that is directly applicable in this framework.

The EPC model is not a consistent PN expansion to 3.5 PN order of the PC model. Instead, it adds 3.5 PN order corrections through a mapping between the **TaylorF2** and the PC models. Such a mapping will necessarily neglect high-order eccentricity-dependent terms at first and

¹ A term of Nth PN order is one that is proportional to $(v/c)^{2N}$, where v is the orbital velocity [62].

higher PN order. We will show, however, that the EPC model is remarkably simple and sufficient for data analysis explorations. For simplicity and consistency with prior work, we adopt the restricted PN (“quadrupole”) approximation, wherein we include the aforementioned PN phase corrections, but we neglect PN amplitude corrections. Recent work suggests that amplitude corrections may play an important role in detection [70–80], but we defer such an analysis to future work.

We compare and validate the EPC model against other waveforms commonly employed in the literature [41, 59]. In particular, we show analytically and numerically that the EPC reduces to the **TaylorF2** model at 3.5 PN order in the limit of zero eccentricity. We also compare the EPC model numerically to other eccentric waveform families for comparable mass binaries [41, 59] using data analysis measures, focusing particularly on the x -model [59]. We find that the EPC model agrees better with the **TaylorF2** 3.5 PN model at small eccentricity than the x -model, primarily because the x -model only includes the dissipative dynamics to 2 PN order. We also find that the EPC model loses accuracy at a slow rate as the eccentricity increases, remaining accurate relative to the x -model up to eccentricities of 0.4.

Once validated, we use the EPC model to study the importance of eccentricity in GW searches with second- and third-generation detectors. We confirm that eccentricity corrections increase the in-band GW signal strength for a fixed mass system, thereby increasing the distances to which the system could be detected. Similarly, the presence of eccentricity increases the range of masses that are accessible to the detectors. We also use the EPC model to estimate the cosmological range to which eccentric inspirals of NSBH and NSNS binaries could be observed. Such binaries have been proposed as possible progenitors of short gamma-ray bursts (SGRBs) [81–85]. Following [86], and assuming that one to a few eccentric binaries form per young massive star cluster over its lifetime, we estimate that aLIGO could observe approximately 0.1–10 events per year out to $z \sim 0.2$, while an array of ET detectors could observe from 60–7900 events per year out to $z \sim 2.3$.

This paper is organized as follows. In Section II we summarize the construction of time-domain and frequency-domain waveform models. Section III presents and develops the EPC model. Section IV studies the astrophysical consequences of doing data analysis with the EPC model. Section V concludes and points to future work. Henceforth, we use geometric units with $G = c = 1$. We follow the conventions in [87] and [61].

II. ECCENTRIC WAVEFORM MODELS

In this section, we review the basics on how to construct time-domain and frequency-domain eccentric waveform models. We concentrate on the time-domain x -model proposed in Ref. [59] and the frequency-domain

PC model proposed in Ref. [61]. Ultimately, detection and parameter estimation is usually carried out in the frequency-domain. By time-domain waveform model, we here mean a waveform that is constructed (usually numerically) in the time-domain and then is DFTed to obtain a frequency representation. By contrast, a frequency-domain model is constructed (usually analytically) directly in the Fourier domain.

A. The x -model

The x -model [59] is a parameter-free, time-domain PN-based waveform family. The binary orbit is given in the Keplerian parameterization to 3 PN order, which can be written as

$$\frac{r}{M} = \frac{1 - e_t \cos u}{x} + \sum_{j=1}^{j=3} \delta r_j x^{j-1}, \quad (1)$$

$$\ell = u - e_t \sin u + \sum_{j=2}^{j=3} \delta \ell_j x^j, \quad (2)$$

while its conservative evolution is given also to 3 PN order by [59]:

$$M \begin{pmatrix} \dot{\phi} \\ \dot{\ell} \end{pmatrix} = \sum_{j=0}^{j=3} \begin{pmatrix} \delta \dot{\phi}_j \\ \delta \dot{\ell}_j \end{pmatrix} x^{3/2+j}. \quad (3)$$

In these equations, r and ϕ are the magnitude of the relative separation vector and the relative orbital phase, e_t is the so-called *temporal* eccentricity, defined by the change in mean anomaly ℓ when the eccentric anomaly u changes by one full cycle to leading PN order, $x = \left(M \dot{\ell}\right)^{2/3} \ll 1$ is the PN expansion parameter, with $\dot{\ell} = n$ the mean Keplerian orbital frequency, $M = m_1 + m_2$ is the total mass, and n the mean motion, and the PN coefficients $(\delta r_j, \delta \ell_j, \delta \dot{\phi}_j, \delta \dot{\ell}_j)$ can be found in [59] and references therein. The above orbital evolution is *conservative* in that energy and (z-component of) angular momentum are conserved quantities, and the equations can thus be derived from a given PN Hamiltonian [88].

The true inspiral evolution, however, is not conservative because GWs carry energy and angular momentum away from the binary. The loss of energy and angular momentum can be mapped to a change in the PN expansion parameter x and the eccentricity e_t , which are no longer conserved, but rather evolved according to the 2 PN equations

$$M \begin{pmatrix} \dot{x} \\ \dot{e}_t \end{pmatrix} = \sum_{j=0}^4 \begin{pmatrix} \delta \dot{x}_j \\ \delta \dot{e}_j \end{pmatrix} x^{5+j/2}, \quad (4)$$

where the dissipation coefficients $(\delta \dot{x}_j, \delta \dot{e}_j)$ can be found for example in [59] and references therein.

The above equations define the orbital evolution in the x -model, which is solved numerically in the time-domain. This evolution has been validated against one numerical relativity simulation of an equal-mass, BH-BH binary with initial eccentricity of 0.1 21 GW cycles before merger [59]. Once the orbital evolution has been obtained, one can obtain the time-domain GW response function to leading (mass-quadrupole or Newtonian) order [89], which one then DFTs to obtain frequency-domain templates for data analysis studies.

The x -model reduces to some well-studied template families used in GW data analysis. For example, in the limit of zero eccentricity, the orbital phase in the x -model reduces to the **TaylorT4** PN model at 2 PN order [40]. In fact, the **TaylorT4** 2 PN differential equations that define the orbital evolution are the same as those of the x -model in the zero-eccentricity limit by construction. However, the amplitude of the x -model differs from 2 PN **TaylorT4** in that $M\dot{\ell}$ in Eq. (3) introduces an additional amplitude contribution [40].

The x -model, however, deviates from some other well-studied template families. One example is the **TaylorF2** templates, a family constructed to model GWs from the quasi-circular inspiral of non-spinning compact binaries. This template family is defined directly in the frequency-domain through the SPA via

$$\tilde{h}(f) = A f^{-7/6} e^{i\Psi_{F2}(f)}, \quad (5)$$

$$\Psi_{F2}(v) = 2\pi f t_c - 2\phi_c - \frac{\pi}{4} + \Psi_{PN}(v), \quad (6)$$

where the PN phase is defined as

$$\Psi_{PN}(v) = \frac{3}{128\eta v^5} \sum_{i=0}^{i=7} a_i v^i, \quad (7)$$

and where $A \propto \mathcal{M}_C^{5/6}/D_L$, with the chirp mass, $\mathcal{M}_C = M\eta^{3/5}$, the symmetric mass ratio $\eta = m_1 m_2 / M^2$, and the luminosity distance D_L , while $v = (\pi M f)^{1/3}$ is the orbital velocity of the binary. The **TaylorF2** waveform phase we will use throughout this article, Eq. (6), includes PN corrections up to 3.5 PN order. The corresponding a_n coefficients in Eq. (6) at this PN order can be found in [69].

The x -model captures all critical features that eccentricity introduces to non-spinning binary physics, both on the dynamics and on the waveform, to high PN order, i.e., to 3 PN order in the conservative dynamics and to 2 PN order in the dissipative dynamics. First and foremost, eccentric binaries precess, and the x -model captures this well at high PN order. Second, eccentricity shortens the duration of the orbit and hence of the waveform, compared to circular binaries starting at the same mean orbital frequency. Third, binaries with eccentricity have complicated, highly modulated waveforms, which is also captured in the x -model to high PN order.

As a concrete example, Fig. 1 shows the waveforms predicted by the x -model for a circular ($e = 0$) and eccentric

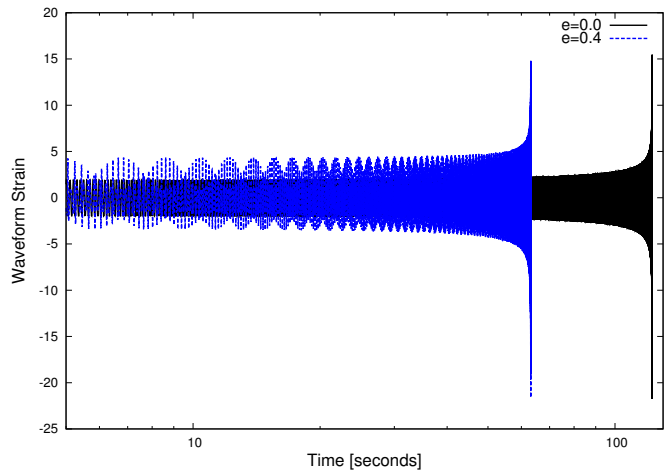


FIG. 1. (Color Online) Sample waveforms for an equal mass binary system with total mass $M = 10M_\odot$ starting at a Keplerian mean orbital frequency $f_K = 5$ Hz. The figure uses a logarithmic scale in the time axis to clearly exhibit the structure of the eccentric waveform at low frequencies.

($e = 0.4$) ($5M_\odot, 5M_\odot$) binary. Observe the amplitude modulations present in the eccentric waveform and the fact that the eccentric inspiral is noticeably shorter. The latter is driven both by long-term effects at low frequencies and by waveform termination. In the figure, the orbital evolution terminates when the system reaches the innermost stable circular orbit (ISCO) for a test particle in an eccentric orbit around a Schwarzschild BH, i.e., $r_{\text{ISCO}} = 6M + 2e_{\text{ISCO}}$ [90], where e_{ISCO} stands for the eccentricity at the ISCO.

B. Limitations of the x -model

Even though the x -model is capable of reproducing the main features of the eccentric numerical simulation used to calibrate it, the model does have some limitations, which we list below:

- *Computational expense.* The x -model requires the numerical solution of the orbital evolution equations in the time-domain at a *small and constant* discretization so that a DFT of the GW response can be accurately computed [41].
- *PN accuracy.* Although the x -model reduces to the 2 PN **TaylorT4** approximant when $e_0 \rightarrow 0$, higher PN order models (e.g. **TaylorT4** at 3.5 PN order) are needed to describe the dynamical evolution of low-mass, quasi-circular binaries at the level of accuracy required for GW data analysis [63].

Let us discuss some of these limitations in more detail. To do so, we employ some basic data analysis tools. Given two signals h_1 and h_2 , the noise-weighted inner

product is defined as

$$(h_1 | h_2) = 2 \int_{f_{\min}}^{f_{\max}} \frac{\tilde{h}_1^*(f) \tilde{h}_2(f) + \tilde{h}_1(f) \tilde{h}_2^*(f)}{S_n(f)} df, \quad (8)$$

and the normalized overlap is

$$(\hat{h}_1 | \hat{h}_2) = \frac{(h_1 | h_2)}{\sqrt{(h_1 | h_1) (h_2 | h_2)}}, \quad (9)$$

where $S_n(f)$ is the power spectral density of the detector noise, and $\tilde{h}(f)$ is the Fourier transform of the signal. For the former, in this paper we always use the Zero Detuned High Power (ZDHP) spectral density for aLIGO [2] and the optical configurations B or D for the Einstein Telescope (ETB, ETD), respectively [5]. The lower limit in integration depends on the detector under consideration, namely $f_{\min} = [1 \text{ Hz}, 10 \text{ Hz}]$, for the ET and aLIGO, respectively, and f_{\max} is the last frequency at which the waveform is sampled.

Maximizing over the time of coalescence, t_c , and the phase of coalesce, ϕ_c , one can compute the maximized overlap, $\mathcal{O}(h_1, h_2)$, between any two given signals, namely

$$\mathcal{O}(h_1, h_2) = \max_{t_c \phi_c} (\hat{h}_1 | \hat{h}_2 e^{i(2\pi f t_c - \phi_c)}) . \quad (10)$$

This quantity is a data analysis measure of how similar two waveforms h_1 and h_2 are, without allowing for any biasing in the system parameters other than t_c and ϕ_c . When quoting overlaps, we will assume optimally oriented sources.

The x -model agrees quite well with the **TaylorT4** waveform family at 2 PN order, but it disagrees with higher PN order families. For example, in the limit of zero eccentricity, the overlap between the x -model and the **TaylorT4** model at 3.5 PN order is roughly 0.5 and 0.7

for binaries with component masses $(1.35M_\odot, 1.35M_\odot)$ and $(6M_\odot, 6M_\odot)$ respectively. This is to be expected, since the x -model is built with 2 PN equations to describe the dissipative dynamics. Even if one compares the x -model to the **TaylorT4** model at 2 PN order, the overlap drops rapidly with increasing eccentricity, crossing the 0.97 threshold at an initial eccentricity at a GW frequency of 10 Hz of approximately 0.02 and 0.05 for binaries with component masses $(1.35M_\odot, 1.35M_\odot)$ and $(6M_\odot, 6M_\odot)$ respectively.

C. The post-Circular Approximation

One of the first attempts to develop a consistent eccentric waveform model in the frequency domain was through the PC approximation [61]; we refer the reader to Sections II and III of [61] for a careful description of the generalization of the SPA to eccentric orbits. The PC stipulates that one can expand all quantities assuming the eccentricity is small. In principle, one can keep an arbitrary number of eccentricity corrections, but in the work of [61], terms up to $\mathcal{O}(e^8)$ were kept. Expanding the time-domain response in this way, one can obtain an expression that is amenable to the SPA when computing the Fourier transform. The main result of the PC approach is the frequency-domain response function for eccentric binary inspirals [61]:

$$\tilde{h}(f > 0) = \mathcal{A} f^{-7/6} \sum_{\ell=1}^{10} \xi_\ell \left(\frac{\ell}{2}\right)^{2/3} e^{-i(\pi/4 + \Psi_\ell)}, \quad (11)$$

where \mathcal{A} is a function of the component masses and the distance to the source only and ξ_ℓ are functions of the beam pattern functions and the eccentricity [61]. The Fourier phase to leading PN order is given by

$$\begin{aligned} \Psi_\ell = & -\ell \phi_c + 2\pi f t_c + \frac{3}{128(M\pi f)^{5/3}} \left(\frac{\ell}{2}\right)^{8/3} \left[1 - \frac{2355}{1462} e_0^2 \chi^{-19/9} + \left(\frac{5222765}{998944} \chi^{-38/9} - \frac{2608555}{444448} \chi^{-19/9} \right) e_0^4 \right. \\ & + \left(-\frac{75356125}{3326976} \chi^{-19/3} - \frac{1326481225}{101334144} \chi^{-19/9} + \frac{17355248095}{455518464} \chi^{-38/9} \right) e_0^6 \\ & + \left(-\frac{250408403375}{1011400704} \chi^{-19/3} + \frac{4537813337273}{39444627456} \chi^{-76/9} - \frac{6505217202575}{277250217984} \chi^{-19/9} + \frac{128274289063885}{830865678336} \chi^{-38/9} \right) e_0^8 \\ & \left. + \mathcal{O}(e_0^{10}) \right], \end{aligned} \quad (12)$$

where $\chi \equiv F/F_0 = f/f_0$, with F the orbital frequency, f the GW frequency and subscript zero to indicate the initial values of these quantities at which the eccentricity e_0 is defined. Such a model does contain amplitude corrections relative to its circular counterpart, because

the ξ_ℓ 's are functions of eccentricity, which in turn is a function of the orbital frequency.

The PC formalism is appealing but the waveforms presented above are not adequate for a data analysis study. This is simply because the expansions have been trun-

cated at leading PN order. Therefore, by construction, the PC model does not reduce to the **TaylorF2** waveform family in the limit of zero eccentricity beyond leading PN order. The resulting overlaps are thus terrible. Having said that, the PC formalism is still promising because one could in principle include higher-order PN corrections, thus systematically improving the waveform family.

III. THE ENHANCED POST-CIRCULAR WAVEFORM MODEL

Systematic, well-controlled PN expressions for time-domain elliptic orbits are available in the literature [65–67, 91–93]. One could in principle use this work to extend the analysis of [61] to higher PN order. Doing so, however, becomes increasingly difficult with higher PN order, so much so that the analytic expressions become unwieldy and the resultant frequency-domain waveform becomes computationally expensive.

Rather than systematically using those expressions as the basis for a Fourier-domain approximation, this paper adopts a physically-motivated *ansatz* to *extrapolate* the form of $\tilde{h}(f)$ from known behavior in two limits: the high order quasi-circular PN approximation and the leading-order PC approximation. Though lacking some eccentricity-dependent modifications beyond leading PN

order, this ansatz is particularly easy to construct, it captures the leading-order effects of eccentricity, and (as we show later) it is effective at reproducing the results of time-domain calculations like the *x*-model at suitable PN orders.

A. Requirements and Construction

We construct the EPC model with the following requirements:

- To zeroth order in the eccentricity, the model must recover the **TaylorF2** PN waveform at 3.5 PN order.
- To zeroth PN order, the model must recover the PC expansion of [61], including eccentricity corrections up to order $\mathcal{O}(e^8)$.

There are an infinite number of ways in which one can modify the PC expansion of [61] to satisfy these two requirements. We choose to employ the same functional form for the Fourier phase as that used in the **TaylorF2** model at 3.5 PN order, Eq. (6). However, we will use a modified velocity function $v(f) \rightarrow v_{\text{ecc}}(f; e_0)$ calculated by equating Eq. (12) at $\ell = 2$ with Eq. (6) at $i = 0$, which leads to:

$$\begin{aligned} v_{\text{ecc}}(f; e_0) = & \left\{ 1 + \frac{471e_0^2}{1462}\chi^{-19/9} + e_0^4 \left(\frac{521711}{444448}\chi^{-19/9} - \frac{39196333705}{533796714784}\chi^{-38/9} \right) \right. \\ & + e_0^6 \left(\frac{265296245}{101334144}\chi^{-19/9} - \frac{1302494157901715}{243411301941504}\chi^{-38/9} + \frac{6142097676388541753}{2135203940630873088}\chi^{-19/3} \right) \\ & + e_0^8 \left\{ \frac{1301043440515}{277250217984}\chi^{-19/9} - \frac{9626858181465026345}{443982214741303296}\chi^{-38/9} + \frac{20410190578639124245219}{649101997951785418752}\chi^{-19/3} \right. \\ & \left. \left. - \frac{2140356054716884783056259067777}{152020619381675205883794309120}\chi^{-76/9} \right\} \right\} (\pi M f)^{1/3}. \end{aligned} \quad (13)$$

With this at hand, the EPC model is defined as the PC model for $\tilde{h}(f)$ of Eq. (11) but with $\Psi_\ell \rightarrow \bar{\Psi}_\ell$, where

$$\bar{\Psi}_\ell \equiv \text{R}_{e_0 \ll 1} \left[2\pi f t_c - \ell \phi_c - \frac{\pi}{4} + \left(\frac{\ell}{2} \right)^{8/3} \Psi_{\text{PN}}(v_{\text{ecc}}) \right], \quad (14)$$

where $\text{R}[\cdot]$ stands for re-expanding in e_0 up to $\mathcal{O}(e_0^8)$. The different PN expressions for $\bar{\Psi}_2$ are presented in Appendix A.

In summary, the EPC model has some appealing features of the two waveform families taken as reference points (the *x*-model at 2 PN order and the **TaylorF2** model at 3.5 PN order). First, as shown in Figure 2, the phase prescription used to construct the EPC model

is reliable for $e_0 \lesssim 0.6$ for a $(6M_\odot, 6M_\odot)$ system, and for $e_0 \lesssim 0.4$ for a $(1.4M_\odot, 1.4M_\odot)$ system. That is, for initial eccentricities at a GW frequency of 10 Hz below these eccentricity values, the error induced by neglecting the $\mathcal{O}(e_0^{10})$ terms in the EPC model lead to a loss of overlap of less than 3%. Second, like the **TaylorF2** model, the EPC family has the advantage that it is already written analytically in the frequency domain. Therefore, it can be readily used as an efficient and accurate waveform family for future searches of eccentric systems, or to explore the efficiency of current matched-filter algorithms to clearly distinguish instrumental glitches from eccentric signals. Regarding efficiency, we have found that, averaging over 100 iterations, the code we have developed to

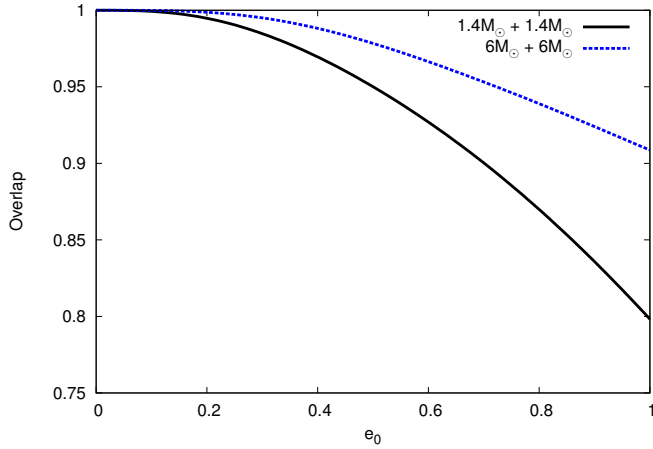


FIG. 2. (Color Online) We explore the accuracy of the phase prescription of the EPC model by deriving two different phase expressions which include eccentricity corrections up to order e^8 and e^6 , respectively. Integrating from a fiducial GW frequency of 10 Hz, the y-axis shows the overlap between both phase approximations. Notice that the overlap between both phase prescriptions is reliable, i.e., the overlap $\gtrsim 0.97$ for $e_0 \lesssim 0.6$ for a $(6M_\odot, 6M_\odot)$ system, and for $e_0 \lesssim 0.4$ for a $(1.4M_\odot, 1.4M_\odot)$ system.

generate EPC waveforms is two times faster than the x -model used in [41] — a numerical code that was enhanced by implementing adaptive time-stepping.

The EPC model also has the advantage of encoding high PN order corrections that faithfully describe the dynamical evolution of quasi-circular binaries and high-order eccentricity corrections that reproduce the dynamics of compact sources with low to moderate values of eccentricity. Of course, this model is not perfect, as we will see next, but it can be systematically improved by correcting the eccentricity-dependent, and higher PN order terms in the phase and amplitude modulations, using the results in [65–67, 91–93].

B. Comparison to Other Models

Figure 3 shows the overlap between different waveform families as a function of GW frequency. We have used 10 Hz and 5 Hz as the initial filtering frequency for aLIGO and ET, respectively. Observe that the EPC model does reduce to the **TaylorF2** 3.5 PN model in the $e_0 \rightarrow 0$ limit. Specifically, the blue curves (**TaylorF2** vs EPC) go to 1 as $e \rightarrow 0$. Observe also that the overlap between the x -model and either the EPC model or the **TaylorF2** 3.5 PN model is roughly the same as $e \rightarrow 0$. Additionally, the x -model is systematically unfaithful to the EPC model by a nearly constant amount, which is nearly independent of eccentricity at each mass. These differences arise due to systematic differences between time- and frequency-domain waveforms, as well as by slight differences in PN

order used to construct these models.

In Figure 3 we also compare the performance of EPC and the x -model against the time domain PN-based **TaylorT4** 3.5 PN approximant, which has been shown to accurately reproduce the features of numerical relativity simulations for the very last few cycles of non-spinning equal mass binary inspirals [63, 69]. Notice that the overlap between EPC and **TaylorT4** 3.5 PN shows that EPC captures faithfully the features of quasi-circular binaries. In contrast, the low-order PN expansion used to construct the x -model leads to substantial drops in overlap in the quasi-circular limit.

Figure 3 also demonstrates the significant impact of eccentricity in data analysis. As also seen in previous studies, the overlap between quasi-circular and eccentric waveforms decreases rapidly as the eccentricity is increased even by a small amount. In addition, the longer a binary spends in band (i.e., the lower the mass or the initial frequency of the system), the greater is the effect of eccentricity on data analysis, since more cycles accumulate.

In principle, higher order eccentricity corrections in the PN expansion that we have neglected could contaminate our ability to identify the unique impact of eccentricity. In fact, the PN approximation for quasi-circular inspirals is known to converge slowly or even diverge, particularly late in the inspiral [12, 94–96]. In practice, however, eccentric effects are weakest at the end of the inspiral and strongest early on, so we expect that eccentric effects are less susceptible to large systematic errors from unknown higher-order PN terms.

IV. ASTROPHYSICS AND COSMOLOGY WITH ECCENTRIC WAVEFORMS

Having constructed a waveform model that captures the main features of eccentric binaries, in this section we compute the signal-to-noise ratio (SNR) distribution of a variety of compact binary sources on eccentric orbits whose mass-ratios represent typical binary BH and NSBH systems. As is well known for circular [68, 97, 98] and eccentric [61] binaries, the presence of multiple harmonics provides additional signal power, particularly at frequencies where the dominant harmonic may be inaccessible. For this reason, eccentric binaries can potentially be detected with very large masses that are inaccessible for quasi-circular inspirals. Due to cosmological redshift, massive eccentric binaries can conceivably be detected to much greater distances than their quasicircular counterparts.

The SNR can be computed via

$$\text{SNR} = \sqrt{\langle h, h \rangle}, \quad (15)$$

where as before, $f_{\text{low}} = 1\text{Hz}$ and 10Hz for ET and aLIGO respectively. Since the PN approximation breaks down as the system approaches merger, we truncate the signal at an orbital frequency corresponding to the inner-

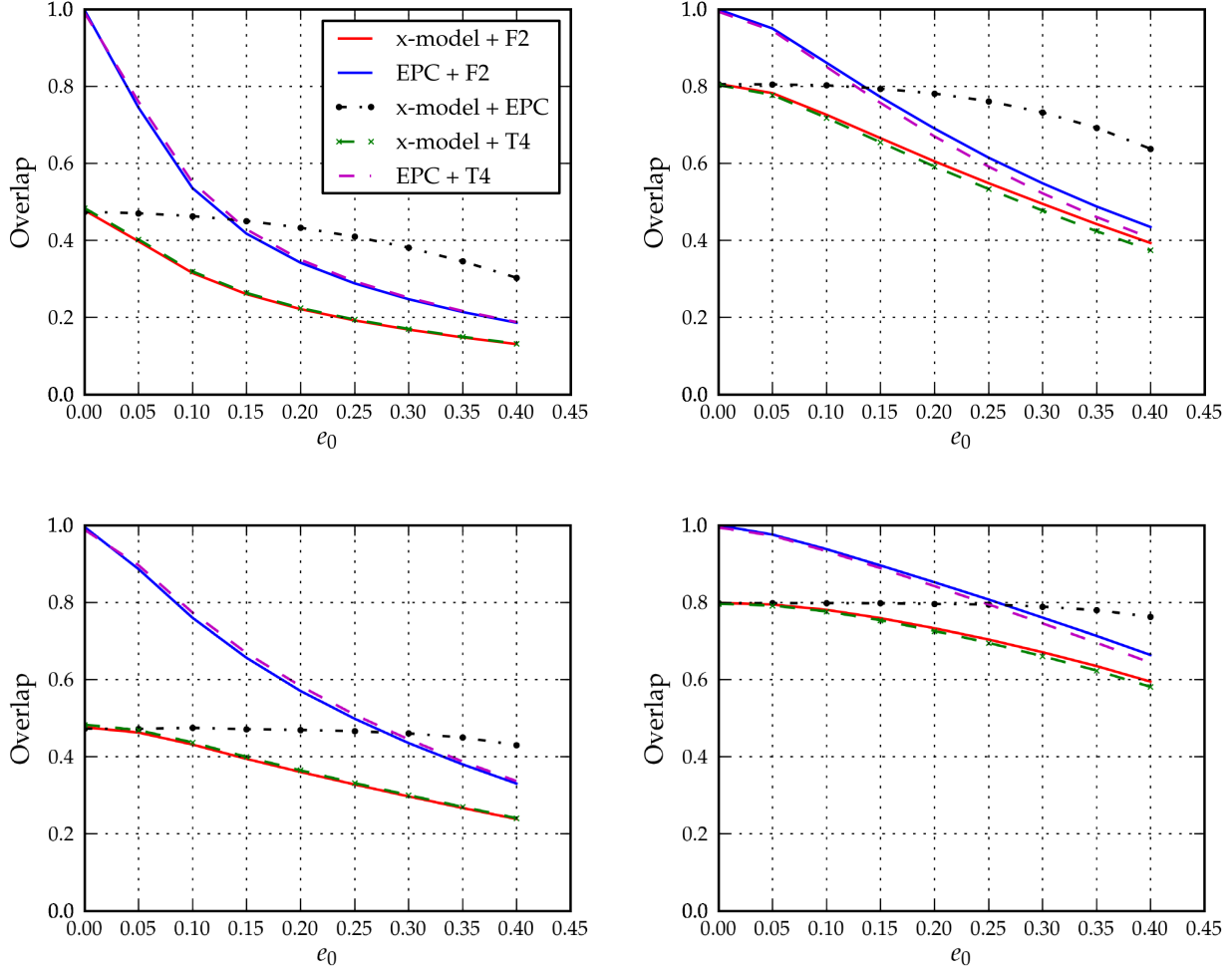


FIG. 3. (Color Online) Overlap between the EPC waveform and three waveform families: **TaylorF2**, **TaylorT4** and the *x*-model; and the *x*-model with **TaylorF2** and **TaylorT4**. Note that both **TaylorF2** and **TaylorT4** include waveform phase corrections at 3.5 PN order (see legend of top-left panel for reference). The different panels show the overlap for binaries with component masses $(1.35M_{\odot}, 1.35M_{\odot})$ (left) and $[(6M_{\odot}, 6M_{\odot})]$ (right). The top panels assume an aLIGO detector with the ZDHP sensitivity configuration and a lower frequency cut-off of 10 Hz, while the bottom panels use an ET-like detector with the ETB configuration and a lower frequency cut-off of 5Hz.

most stable circular orbit (ISCO) of a test particle in Schwarzschild spacetime; eccentricity-corrections to the ISCO are negligible for SNR calculations.

As shown in [71], different GW harmonics will contribute signal over different GW frequency ranges if we terminate the integration of Eq. (15) at the ISCO orbital frequency. Hence, in order to ensure that the harmonics contribute to the SNR within their region of validity, we will truncate the waveforms using step functions, ($H(x) = 1, \text{if } x \geq 0$, and $H(x) = 0$ otherwise), as follows [61]:

$$\tilde{h} = \mathcal{A} f^{-7/6} \sum_{\ell=1}^{\ell=10} \left(\frac{\ell}{2}\right)^{2/3} \xi_{\ell} e^{i\Psi_{\ell}} H(\ell F_{\text{ISCO}} - 2F), \quad (16)$$

where F is the Keplerian mean orbital frequency and

F_{ISCO} is that frequency at ISCO.

A. Increase in Reach

An immediate consequence of the inclusion of eccentricity in the waveform model is the ability to detect systems at a fixed mass farther out, and to detect heavier systems at a fixed distance. The former case was already made in [61] for space-borne GW detectors, so let us discuss the latter. Figure 4 shows the SNR contribution from each harmonic versus total mass for an optimally oriented equal-mass binary directly overhead for a single aLIGO (left) or ET detector (right). The SNR contribu-

tion is obtained from

$$\Delta \text{SNR} = \sqrt{|\text{SNR}_{\ell_{\max}}^2 - \text{SNR}_{\ell_{\max}=10}^2|}, \quad \ell_{\max} = 2, 3, \dots, 9, \quad (17)$$

where the subscript ℓ_{\max} indicates the number of harmonics included in the calculation of SNR. Observe that for the eccentric inspirals investigated here, the role of higher harmonics is important.

Based on Fig. 4, we notice that in the context of aLIGO, the SNR difference peaks at approximately $(200M_{\odot}, 500M_{\odot}, 600M_{\odot}, 700M_{\odot})$ for $\ell_{\max} = 2, 3, 4$ and 5, respectively. In the context of ET, the SNR difference reaches a maximum at approximately $(600M_{\odot}, 4500M_{\odot}, 5500M_{\odot}, 7000M_{\odot})$ for $\ell_{\max} = 2, 3, 4$ and 5, respectively. We have also found that, both for aLIGO and ET, in order to ensure that the SNR difference (see Eq. (17)) is less than 10, then we need to include up to the ninth harmonic throughout the whole parameter space.

Figure 5 shows the SNR versus total mass and eccentricity for an optimally-oriented, equal-mass binary directly overhead for a single detector. We have used two different detectors — aLIGO and ET — to illustrate the increase in reach as a function of sensitivity and eccentricity. We should emphasize, however, that for total mass $M \gtrsim 300M_{\odot}$, Figure 4 provides a conservative estimate of the SNR that may be expected from these type of events, based on well understood physics only. All of these results are consistent with those presented in [61].

B. Eccentric Neutron Star-Black Hole Binaries

Observational and theoretical evidence suggests that SGRBs may be associated with NSNS and NSBH mergers [81, 82, 99]. Our previous calculations show that eccentric mergers can be detected with a larger range than quasicircular mergers.

To assess the astrophysical impact of including eccentricity in event rate calculations, we estimate the detection rate for a simply-parameterized astrophysical toy model, using EPC waveforms. Our astrophysical toy model assumes that one to a few eccentric, merging NSBH binaries form per young massive star cluster, over its lifetime [100]. Following [86], we then extrapolate this optimistic formation rate to the entire universe, assuming that the star formation rate (SFR) per unit volume in the Universe, and the cluster formation rate (CFR) are given by

$$\text{SFR} = 1M_{\odot} \times 10^{-2} \text{ galaxies Mpc}^{-3} \text{ yr}^{-1}, \quad (18)$$

$$\text{CFR} = \text{SFR}/10^6 M_{\odot}. \quad (19)$$

Assuming that a significant fraction of star formation occurs in clusters, g_{cl} , and that a substantial fraction of clusters undergo evaporation and segregation, g_{evap} , then present-merger rates of compact binaries can be approximated as $n_{\text{rate}} = \text{CFR} \times \text{number per cluster}$:

$$n_{\text{rate}} \approx 10^{-2} g_{\text{evap}} g_{\text{cl}} \text{ Mpc}^{-3} \text{ Myr}^{-1}. \quad (20)$$

Given the lack of a universally accepted model for star formation history, we will present results for merger rates normalized by the factor $\Gamma = g_{\text{evap}} g_{\text{cl}}$, i.e.,

$$n_{\text{rate}} = 10 \Gamma \text{ Gpc}^{-3} \text{ yr}^{-1}. \quad (21)$$

Using Eq. (21), we notice that we can readily estimate the number of present-day event rates using the relation

$$n_{\text{events}} = 10 \Gamma \left(\frac{V_c}{\text{Gpc}^3} \right) \text{ yr}^{-1}. \quad (22)$$

If one assumes that 10% of all clusters survive disruption due to photoionization and supernova gas-driven ejection during their first ~ 10 Myr of existence, then $\Gamma \sim 5 \times 10^{-2}$. Under more optimistic assumptions, Γ can plausibly be as high as ~ 1 .

For a given network of identical GW detectors, a straightforward calculation provides the detection volume V_c for a given source [101, 102], including cosmology. Specifically, for each source, sky location, and distance D_L , we evaluate the SNR in each detector and hence the network SNR ρ , carefully accounting for the topology and geographical location of its components [101]. For each distance and sky position, a fraction of sources have $\rho > \rho_{\text{threshold}} \equiv 10$. Using concordance cosmology, we translate the SNR of sources with $\rho > \rho_{\text{threshold}}$ to a detection range, and finally into a redshift estimate, which in turn determines the detection volume V_c .

Network	LIGO Livingston	LIGO Hanford	Virgo	INDIGO
C_1	✓			
C_2	✓	✓		
C_3	✓	✓	✓	
C_4	✓	✓	✓	✓

TABLE I. Configurations, C_i , used to compute the SNR distributions of compact binaries that will be targeted by second and third generation ground-based detectors.

We evaluate the detection volume for several combinations of detectors, as described in Table I. The first detector network is comprised of four LIGO-type L-shaped detectors that operate with the target ZDHP laser configuration, from a low frequency cut-off of 10Hz. The second detector network consists of up to four ET type detectors, including one triangular-shaped detector at the geographic location of Virgo, and up to three L-shaped detectors at the location of LIGO Livingston, LIGO Hanford, and the proposed location for LIGO India (see Table II). We explore two different design sensitivities for these networks, ETB and ETD, and we assume that both configurations operate from a low frequency cut-off of 1Hz. Given that the results obtained by using ETB and ETD are quantitatively similar, we will only quote results derived using ETB in the following. Table II provides the comoving volume for several compact binaries that may be detected during the advanced detector era and beyond using a four detector network, assuming a uniform

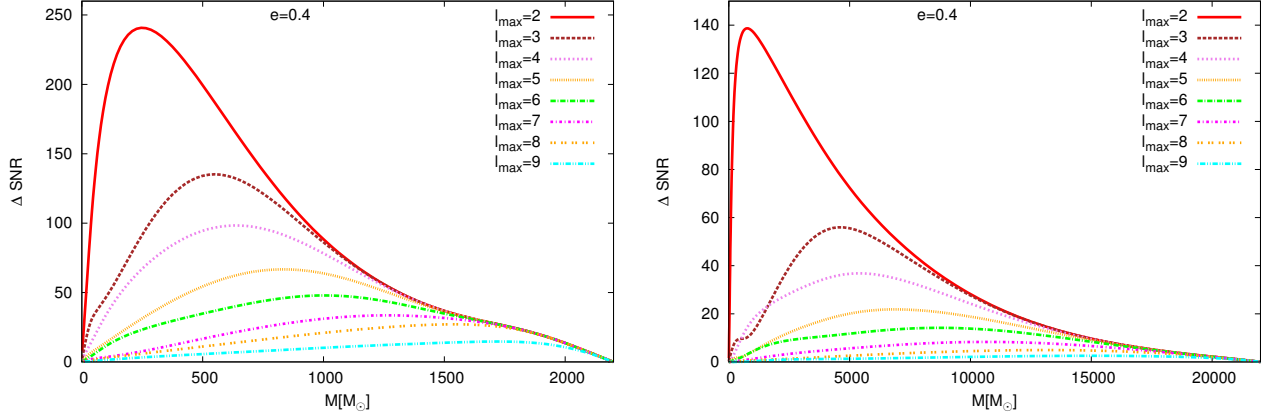


FIG. 4. (Color Online) Absolute contribution from each harmonic to the total SNR of an optimally oriented equal-mass binary directly overhead a single detector (see Eq. (17) for further reference). The left Figure assumes a LIGO-type detector and a fixed distance of $D_L = 100$ Mpc. The right Figure assumes an ET-type detector and a fixed distance of $D_L = 1$ Gpc. We have assumed that the binaries have an initial eccentricity $e_0 = 0.4$ at a GW frequency of 1 Hz for ET and 10 Hz for aLIGO. For comparison, Fig. 5 shows the total SNR versus mass.

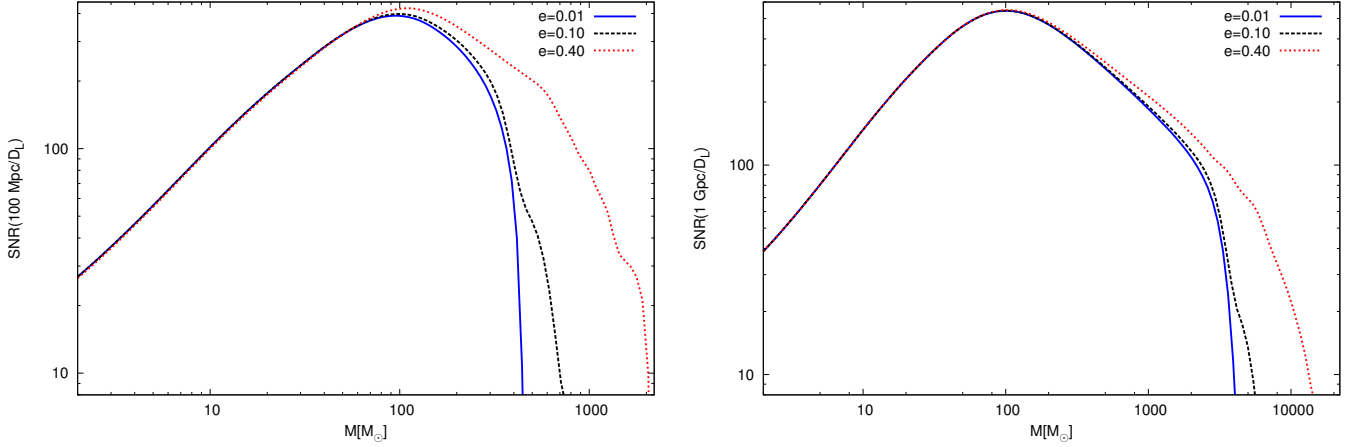


FIG. 5. (Color Online) SNR versus eccentricity for an optimally-oriented, equal-mass binary directly overhead a single detector. Observe that eccentricity can significantly increase the reach of instruments, if all power involved can be captured. For the LIGO-type detector, the SNR is normalized at a distance $D_L = 100$ Mpc, while for the ET detector we use the ETB configuration and normalize the SNR at $D_L = 1$ Gpc.

distribution of inclination angles and adopting an eccentricity $e_0 = 0.4$ at $f_{\text{orb}} = 0.5$ Hz and 5 Hz for ET and aLIGO, respectively. For context, these tables also provide the median luminosity distance and associated redshift. Based on Table II, we notice that in a conservative or in an optimistic scenario ($\Gamma \sim 5 \times 10^{-2}$ or 1), an array of four advanced GW detectors could detect from 0.1 to 10 events per year. In Figure 6 we present the redshift distribution up to which these events may be detected depending on the number of operating GW detectors. Notice that some of these events could be detected up to cosmological redshifts $z \sim 0.2$.

We have found that the SNR distribution of eccentric inspirals is quantitatively similar to that obtained for quasi-circular binaries, even though the former stay in

the sensitive frequency band of GW detectors only half of the time the latter do. In other words, assuming optimal filtering, the event rate does not increase or decrease significantly due to the presence of eccentricity. Moreover, prior investigations have showed that existing searches will only be slightly selection biased against eccentric binaries with $e_0 \lesssim 0.4$. Hence, for existing searches, the relative detection rate for eccentric and quasi-circular binaries is effectively identical. By contrast, our work shows that eccentricity produces a significant, measurable effect on radiated GWs, unless $e_0 \lesssim 10^{-2}$. Measurement of eccentricity thereby provides a mechanism to help distinguish between different astrophysical formation scenarios for merging binaries [8, 10, 14, 103].

aLIGO				
D_L [Gpc]	z	m [M_\odot]	M [M_\odot]	V_c [Gpc^3]
0.425	0.090	1.38	5.50	0.255
0.612	0.126	1.33	10.66	0.682
0.661	0.135	1.41	14.98	0.834
0.726	0.147	1.39	20.05	1.066
ET				
D_L [Gpc]	z	m [M_\odot]	M [M_\odot]	V_c [Gpc^3]
15.168	1.917	1.71	5.14	589.596
17.587	2.164	1.26	9.48	719.221
18.894	2.296	1.51	14.56	789.238
16.593	2.063	1.31	19.26	665.925

TABLE II. Average range (D_L), redshift (z), source-frame masses [M , m], and co-moving volume (V_c), for several combinations of compact binary systems uniformly distributed in the sky with initial eccentricity $e_0 = 0.4$ at a Keplerian mean orbital frequency of 0.5 Hz and 5 Hz for ETB and aLIGO respectively, and using the networks described in Table I with aLIGO's ZDHP (top) and the ETB (bottom) noise configurations. These results are obtained by fixing the masses and initial eccentricity of the sources, and then running Monte Carlo simulations over random choices of the extrinsic parameters: inclination angles, location of the source in the sky, polarization angle, etc.

C. Third generation detectors

In the context of third generation detectors, Table II shows that an array of four ET-like detectors will enable the detection of NSBH mergers up to redshifts $z \sim 2.3$. Therefore, detection of gravitational radiation emitted by high redshift NSBH mergers could be used in conjunction with optical observations to provide further insight on the astrophysical mechanisms that lead to the formation of high-redshift SGRBs.

If we repeat our earlier exercise, now using Table II in Eq. (22) along with the conservative or the optimistic scenario ($\Gamma \sim 5 \times 10^{-2}$ or 1), we find that an array of four third generation GW detectors could observe from 60 to 7900 NSBH mergers per year up to redshift $z \sim 2.3$, again depending on the components masses under consideration (see Figure 6).

The results presented in this final Section suggest that a network of aLIGO/VIRGO type detectors operating at design sensitivity will be capable of detecting NSBH mergers occurring at cosmological distances, and could enable a joint search of GW sources and their electromagnetic counterparts. Looking ahead, our results also suggest that third generation GW detectors may enable us to discover inspiraling eccentric NSBH binaries at redshifts comparable to the most distant known SGRBs.

V. CONCLUSIONS

In this paper we have developed and studied a fast and accurate ready-to-use waveform model (the EPC model) to search for compact binary mergers with significant ec-

centricity ($e < 0.4$) just prior to merger [7–10]. We have shown that the EPC model includes desirable features of higher-order PN expansions for quasi-circular waveforms, and also includes the same dynamical contributions as time-domain PN-based eccentric waveforms that have been successfully compared with results from numerical relativity.

Anticipating that matched filtering in the advanced detection era and beyond may be carried out in the frequency domain, the EPC model has been developed in the frequency domain using the SPA. This model is a natural extension of the frequency domain quasicircular PN approximant **TaylorF2** 3.5 PN to now include eccentricity. The EPC model could be used to develop an optimal search that targets moderately eccentric events, which may be important sources for advanced and third generation GW detectors [2–5], and complementary missions such as NANOGrav [104, 105] and space-based missions such as eLISA [106, 107].

We have explored the astrophysics that could be studied if eccentric mergers occur in Nature by computing the improvement in detection range of second and third generation GW detectors. We have also carried out Monte Carlo simulations to explore the SNR distribution of a population of eccentric sources uniformly distributed in the sky, assuming the existence of two types of networks: (i) an array of up to four second generation detectors at the geographical locations of existing and planned LIGO-type detectors, which operate at the target sensitivity of the ZDHP configuration from a low frequency cut-off of 10 Hz, and (ii) an array of up to four ET-type detectors at the geographical locations of existing and planned GW detectors, which operate at the target sensitivity of the ETB and ETD configurations from a low frequency cut-off of 1 Hz. Using the median of the SNR distribution of compact binaries whose source-frame masses represent typical NSBH systems, we have found that GW observations will enable us to observe the merger of eccentric NSBH systems up to redshifts $z \sim 0.2$ and $z \sim 2.3$, in the context of second and third generation detectors, respectively. These results suggest that a detector network in the advanced detector era may be capable of testing whether the compact object merger model is the correct description for the generation of SGRBs at cosmological distances [108, 109].

The estimates presented here may be refined once we take into account the SNR that is generated during the merger and ringdown phases of NSBH systems. In order to have a complete picture of the dynamical evolution of these events, we may need to develop a complete inspiral-merger-ringdown model in a similar manner as has been done for quasi-circular binaries with comparable and intermediate mass-ratios [110–114]. Moreover, the EPC model itself may be improved by consistently accounting for PN corrections to eccentric waveforms in the PC approximation. Such improvements would not only enter in the Fourier phase, but also in relaxing the restricted PN approximation to account for PN amplitude corrections.

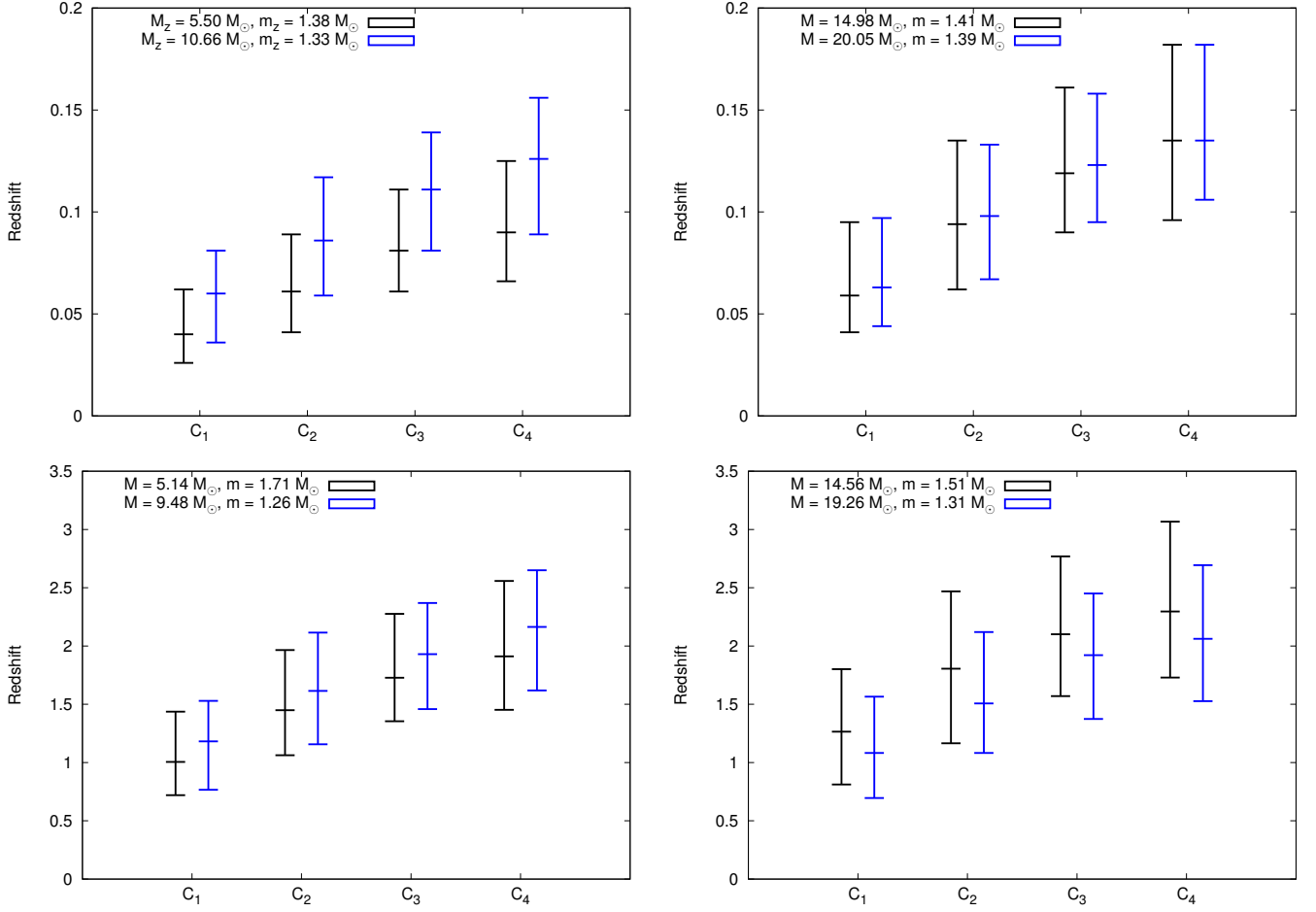


FIG. 6. (Color Online) We run Monte Carlo simulations to estimate the SNR distributions of typical NSBH mergers assuming uniform distributions in the extrinsic parameters of the sources: inclination angles, location of the source in the sky, polarization angle, etc. Using concordance cosmology, we translated the median of the SNR distribution to a detection range, and finally into a redshift estimate. The error bars represent the upper and lower quartiles of the redshift distributions. Each panel shows the redshift distribution for an array of aLIGO detectors — top panels— and ET detectors — bottom panels— with the sensitivity of the target [ETB, ZDHP] configuration operating from a low frequency cut-off of [1 Hz, 10 Hz], respectively. The x axis indicates the number of detectors in the array, as defined in Table I.

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Appendix A: Finite mass-ratio corrections in the EPC model

In this Appendix, we provide the PN corrections introduced to the EPC model. Using the convention

$$\bar{\Psi}_2(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{128\eta(\pi M f)^{5/3}} \sum_{n=0}^7 \alpha_n (\pi M f)^{n/3}, \quad (\text{A1})$$

and $x = (\pi M f)^{1/3}$, the α_n coefficients are given by:

$$\alpha_0 = 1 + \frac{6505217202575e_0^8}{277250217984\chi^{19/9}} - \frac{34527299926564117885e_0^8}{443982214741303296\chi^{38/9}} + \frac{70854027727533517312175e_0^8}{649101997951785418752\chi^{19/3}} \quad (\text{A2})$$

$$- \frac{1480771978371648379295884378817e_0^8}{30404123876335041176758861824\chi^{76/9}} + \frac{1326481225e_0^6}{101334144\chi^{19/9}} - \frac{4671472830986095e_0^6}{243411301941504\chi^{38/9}} + \frac{21322307471421461725e_0^6}{2135203940630873088\chi^{19/3}}$$

$$+ \frac{2608555e_0^4}{444448\chi^{19/9}} - \frac{1405799828765e_0^4}{533796714784\chi^{38/9}} + \frac{2355e_0^2}{1462\chi^{19/9}},$$

$$\alpha_2 = \frac{71557389228325e_0^8\eta}{831750653952\chi^{19/9}} - \frac{454638749586390872855e_0^8\eta}{1331946644223909888\chi^{38/9}} + \frac{942278545508138530341925e_0^8\eta}{1947305993855356256256\chi^{19/3}} \quad (\text{A3})$$

$$- \frac{19670327207880872553538752792427e_0^8\eta}{91212371629005123530276585472\chi^{76/9}} + \frac{690482340216175e_0^8}{9981007847424\chi^{19/9}} - \frac{30708780994789856230115e_0^8}{111883518114808430592\chi^{38/9}}$$

$$+ \frac{63646632664776993458550025e_0^8}{163573703483849925525504\chi^{19/3}} - \frac{1328641192314135300661753938615751e_0^8}{7661839216836430376543233179648\chi^{76/9}} + \frac{14591293475e_0^6\eta}{304002432\chi^{19/9}}$$

$$- \frac{61511689912720685e_0^6\eta}{730233905824512\chi^{38/9}} + \frac{283562607736424474975e_0^6\eta}{6405611821892619264\chi^{19/3}} + \frac{985575550175e_0^6}{25536204288\chi^{19/9}} - \frac{4154835055013769905e_0^6}{61339648089259008\chi^{38/9}}$$

$$+ \frac{19153365231651216809675e_0^6}{538071393038980018176\chi^{19/3}} + \frac{28694105e_0^4\eta}{1333344\chi^{19/9}} - \frac{18510890735095e_0^4\eta}{1601390144352\chi^{38/9}} + \frac{1938156365e_0^4}{112000896\chi^{19/9}} - \frac{1250326528743235e_0^4}{134516772125568\chi^{38/9}}$$

$$+ \frac{8635e_0^2\eta}{1462\chi^{19/9}} + \frac{583255e_0^2}{122808\chi^{19/9}} + \frac{55\eta}{9} + \frac{3715}{756},$$

$$\alpha_3 = -\frac{1301043440515\pi e_0^8}{8664069312\chi^{19/9}} + \frac{8946508632426975653\pi e_0^8}{13874444210665728\chi^{38/9}} - \frac{18731723667094131800179\pi e_0^8}{20284437435993294336\chi^{19/3}} \quad (\text{A4})$$

$$+ \frac{115205353044886385251076880001\pi e_0^8}{279449667981020599051092480\chi^{76/9}} - \frac{265296245\pi e_0^6}{3166692\chi^{19/9}} + \frac{1210444259975591\pi e_0^6}{7606603185672\chi^{38/9}} - \frac{5636991774629591273\pi e_0^6}{66725123144714784\chi^{19/3}}$$

$$- \frac{521711\pi e_0^4}{13889\chi^{19/9}} + \frac{364262491717\pi e_0^4}{16681147337\chi^{38/9}} - \frac{7536\pi e_0^2}{731\chi^{19/9}} - 16\pi,$$

$$\alpha_4 = \frac{4013719013988775e_0^8\eta^2}{19962015694848\chi^{19/9}} - \frac{29698857489819606274325e_0^8\eta^2}{31966719461373837312\chi^{38/9}} + \frac{62965437935101698296500615e_0^8\eta^2}{46735343852528550150144\chi^{19/3}} \quad (\text{A5})$$

$$- \frac{1320599685760317911145711844818409e_0^8\eta^2}{2189096919096122964726638051328\chi^{76/9}} + \frac{5045260598968525e_0^8\eta}{19962015694848\chi^{19/9}} - \frac{261321065335868140135025e_0^8\eta}{223767036229616861184\chi^{38/9}}$$

$$+ \frac{554034623257159027636469755e_0^8\eta}{327147406967699851051008\chi^{19/3}} - \frac{11619993021057967487212430478961333e_0^8\eta}{15323678433672860753086466359296\chi^{76/9}} + \frac{2842476030950240425e_0^8}{20121711820406784\chi^{19/9}}$$

$$- \frac{147227055972380882628700925e_0^8}{225557172519453796073472\chi^{38/9}} + \frac{312140494238689330362483671935e_0^8}{329764586223441449859416064\chi^{19/3}}$$

$$- \frac{6546649274949058130045037091614679921e_0^8}{15446267861142243639111158090170368\chi^{76/9}} + \frac{818438915825e_0^6\eta^2}{7296058368\chi^{19/9}} - \frac{4018194477126790775e_0^6\eta^2}{17525613739788288\chi^{38/9}}$$

$$+ \frac{18948371331658651308005e_0^6\eta^2}{153734683725422862336\chi^{19/3}} + \frac{7201466570525e_0^6\eta}{51072408576\chi^{19/9}} - \frac{35356203916242053675e_0^6\eta}{122679296178518016\chi^{38/9}} + \frac{166727241425566965885185e_0^6\eta}{1076142786077960036352\chi^{19/3}}$$

$$+ \frac{4057272307914425e_0^6}{51480987844608\chi^{19/9}} - \frac{19919518567158561620975e_0^6}{123660730547946160128\chi^{38/9}} + \frac{93933341630661850846368845e_0^6}{1084751928366583716642816\chi^{19/3}} + \frac{1609478435e_0^4\eta^2}{32000256\chi^{19/9}}$$

$$- \frac{1209206884479925e_0^4\eta^2}{38433363464448\chi^{38/9}} + \frac{14161845095e_0^4\eta}{224001792\chi^{19/9}} - \frac{10639844693422225e_0^4\eta}{269033544251136\chi^{38/9}} + \frac{7978716747515e_0^4}{225793806336\chi^{19/9}}$$

$$\alpha_5 = -\frac{5994438328967367325e_0^4}{271185812605145088\chi^{38/9}} + \frac{484345e_0^2\eta^2}{35088\chi^{19/9}} + \frac{4261765e_0^2\eta}{245616\chi^{19/9}} + \frac{2401058305e_0^2}{247580928\chi^{19/9}} + \frac{3085\eta^2}{72} + \frac{27145\eta}{504} + \frac{15293365}{508032} \\ - \frac{84567823633475\pi e_0^8\eta}{831750653952\chi^{19/9}} + \frac{669968502482700007405\pi e_0^8\eta}{1331946644223909888\chi^{38/9}} - \frac{1440902965169982699005075\pi e_0^8\eta}{1947305993855356256256\chi^{19/3}} \quad (A6)$$

$$+ \frac{2334216112662079584736091244017\pi e_0^8\eta}{7016336279154240271559737344\chi^{76/9}} + \frac{7182689108386025\pi e_0^8}{9981007847424\chi^{19/9}} - \frac{56903148963613058870695\pi e_0^8}{15983359730686918656\chi^{38/9}} \\ + \frac{122381747448338420666046425\pi e_0^8}{23367671926264275075072\chi^{19/3}} - \frac{18041156334765213110425249225007393\pi e_0^8}{7661839216836430376543233179648\chi^{76/9}} - \frac{17244255925\pi e_0^8\eta}{304002432\chi^{19/9}} \\ + \frac{90645363628809535\pi e_0^6\eta}{730233905824512\chi^{38/9}} - \frac{433615096349678814025\pi e_0^6\eta}{6405611821892619264\chi^{19/3}} + \frac{10252373388025\pi e_0^6}{25536204288\chi^{19/9}} - \frac{7698879291066691165\pi e_0^6}{8762806869894144\chi^{38/9}} \\ + \frac{36828693183369973116475\pi e_0^6}{76867341862711431168\chi^{19/3}} - \frac{33911215\pi e_0^4\eta}{1333344\chi^{19/9}} + \frac{27278171420045\pi e_0^4\eta}{1601390144352\chi^{38/9}} + \frac{20161521595\pi e_0^4}{112000896\chi^{19/9}} - \frac{2316846009950855\pi e_0^4}{19216681732224\chi^{38/9}} \\ - \frac{10205\pi e_0^2\eta}{1462\chi^{19/9}} + \frac{6067265\pi e_0^2}{122808\chi^{19/9}} - \frac{65\pi\eta}{9} - \frac{65}{3}\pi\eta \log(\sqrt{6}x) + \frac{38645}{252}\pi \log(\sqrt{6}x) + \frac{38645\pi}{756}, \quad (A7)$$

$$\alpha_6 = \frac{19887378305015 \log(4x)e_0^8}{12996103968\chi^{19/9}} - \frac{1175668628910900667003 \log(4x)e_0^8}{145681664211990144\chi^{38/9}} + \frac{2568467130783058548836609 \log(4x)e_0^8}{212986593077929590528\chi^{19/3}} \\ - \frac{271831420459973064883705332651019 \log(4x)e_0^8}{49881765734612176930620007680\chi^{76/9}} + \frac{166305877783829875\eta^3 e_0^8}{359316282507264\chi^{19/9}} - \frac{98950858868368325\eta^2 e_0^8}{479088376676352\chi^{19/9}} \\ + \frac{19887378305015\gamma e_0^8}{12996103968\chi^{19/9}} - \frac{2933852958361325\pi^2\eta e_0^8}{3327002615808\chi^{19/9}} + \frac{2925073821111304814575\eta e_0^8}{120730270922440704\chi^{19/9}} + \frac{6505217202575\pi^2 e_0^8}{6498051984\chi^{19/9}} \\ - \frac{487482116356903781533459e_0^8}{37184923444111736832\chi^{19/9}} - \frac{1404484509257344651959425\eta^3 e_0^8}{575400950304729071616\chi^{38/9}} + \frac{15767144696141518291115\eta^2 e_0^8}{14475495605150416896\chi^{38/9}} \\ - \frac{1175668628910900667003\gamma e_0^8}{145681664211990144\chi^{38/9}} + \frac{24776941665365243028895\pi^2\eta e_0^8}{5327786576895639552\chi^{38/9}} - \frac{172919601366556448542601792915\eta e_0^8}{1353343035116722776440832\chi^{38/9}} \\ - \frac{54937786397705638645\pi^2 e_0^8}{10405833157999296\chi^{38/9}} + \frac{145132197850314788879078394300739e_0^8}{2084148274079753075718881280\chi^{38/9}} + \frac{3068358046657424850514388275\eta^3 e_0^8}{841236189345513902702592\chi^{19/3}} \\ - \frac{34446264791342888014771345\eta^2 e_0^8}{21163174574729909501952\chi^{19/3}} + \frac{2568467130783058548836609\gamma e_0^8}{212986593077929590528\chi^{19/3}} - \frac{54129844672110252594640685\pi^2\eta e_0^8}{7789223975421425025024\chi^{19/3}} \\ + \frac{377775081733314667944614526636745\eta e_0^8}{1978587517340648699156496384\chi^{19/3}} + \frac{120021828541264418169935\pi^2 e_0^8}{15213328076994970752\chi^{19/3}} \\ - \frac{317665040175671089647133596190236377e_0^8}{3047024776704598996701004431360\chi^{19/3}} - \frac{64947385645413190689270344198348605\eta^3 e_0^8}{39403744543730213365079484923904\chi^{76/9}} \\ + \frac{729117922168338968426387200662079\eta^2 e_0^8}{991289170911451908555458740224\chi^{76/9}} - \frac{271831420459973064883705332651019\gamma e_0^8}{49881765734612176930620007680\chi^{76/9}} \\ + \frac{1145756734835961236098608458183267\pi^2\eta e_0^8}{364849486516020494121106341888\chi^{76/9}} - \frac{7996297546594766343088603842357850964359\eta e_0^8}{92677607166853461834666948541022208\chi^{76/9}} \\ - \frac{2540480565046477241903788155617\pi^2 e_0^8}{712596653351602527580285824\chi^{76/9}} + \frac{33641288006290009112960754500321337034087507e_0^8}{713617575184771656126935503765871001600\chi^{76/9}} \\ + \frac{28386698215 \log(4x)e_0^6}{33250266\chi^{19/9}} - \frac{159065553051674041 \log(4x)e_0^6}{79869333449556\chi^{38/9}} + \frac{772936241583827429683 \log(4x)e_0^6}{700613793019505232\chi^{19/3}} + \frac{33911492517125\eta^3 e_0^6}{131329050624\chi^{19/9}} \\ - \frac{20177105913475\eta^2 e_0^6}{175105400832\chi^{19/9}} + \frac{28386698215\gamma e_0^6}{33250266\chi^{19/9}} - \frac{598243032475\pi^2\eta e_0^6}{1216009728\chi^{19/9}} + \frac{4175170127655540575\eta e_0^6}{308885927067648\chi^{19/9}} + \frac{1326481225\pi^2 e_0^6}{2375019\chi^{19/9}} \\ - \frac{695818599616190365379e_0^6}{95136865536835584\chi^{19/9}} - \frac{190023872138600320475\eta^3 e_0^6}{315461047316189184\chi^{38/9}} + \frac{2133262323636936905\eta^2 e_0^6}{7936126976507904\chi^{38/9}} - \frac{159065553051674041\gamma e_0^6}{79869333449556\chi^{38/9}} \\ + \frac{3352269365715186565\pi^2\eta e_0^6}{2920935623298048\chi^{38/9}} - \frac{23395667238587898146461505\eta e_0^6}{741964383287676960768\chi^{38/9}} - \frac{7432969768769815\pi^2 e_0^6}{5704952389254\chi^{38/9}} \\ + \frac{19636088561834767348202868433e_0^6}{1142625150263022519582720\chi^{38/9}} + \frac{923369860564978889712425\eta^3 e_0^6}{2767224307057611522048\chi^{19/3}} - \frac{10366014081054134220515\eta^2 e_0^6}{69615705837927333888\chi^{19/3}}$$

$$\begin{aligned}
& + \frac{772936241583827429683\gamma e_0^6}{700613793019505232\chi^{19/3}} - \frac{16289450698799353775095\pi^2\eta e_0^6}{25622447287570477056\chi^{19/3}} + \frac{113684947858355301819023330315\eta e_0^6}{6508511570199502299856896\chi^{19/3}} \\
& + \frac{36118515961861094845\pi^2 e_0^6}{50043842358536088\chi^{19/3}} - \frac{95595859216271769379215647363899e_0^6}{10023107818107233541779619840\chi^{19/3}} + \frac{111646154\log(4x)e_0^4}{291669\chi^{19/9}} \\
& - \frac{95736113783734\log(4x)e_0^4}{350304094077\chi^{38/9}} + \frac{66687708575\eta^3 e_0^4}{576004608\chi^{19/9}} - \frac{39678730105\eta^2 e_0^4}{768006144\chi^{19/9}} + \frac{111646154\gamma e_0^4}{291669\chi^{19/9}} - \frac{1176458305\pi^2\eta e_0^4}{5333376\chi^{19/9}} \\
& + \frac{8210565447201485\eta e_0^4}{1354762838016\chi^{19/9}} + \frac{10434220\pi^2 e_0^4}{41667\chi^{19/9}} - \frac{6841714201879530781e_0^4}{2086334770544640\chi^{19/9}} - \frac{57184433385073825\eta^3 e_0^4}{691800542360064\chi^{38/9}} \\
& + \frac{641968800372235\eta^2 e_0^4}{17403787229184\chi^{38/9}} - \frac{95736113783734\gamma e_0^4}{350304094077\chi^{38/9}} + \frac{1008808114870655\pi^2\eta e_0^4}{6405560577408\chi^{38/9}} - \frac{7040525801561209192435\eta e_0^4}{1627114875630870528\chi^{38/9}} \\
& - \frac{8947300353620\pi^2 e_0^4}{50043442011\chi^{38/9}} + \frac{5909144917795596553777571e_0^4}{2505756908471540613120\chi^{38/9}} + \frac{537568\log(4x)e_0^2}{5117\chi^{19/9}} + \frac{20068525\eta^3 e_0^2}{631584\chi^{19/9}} - \frac{11940635\eta^2 e_0^2}{842112\chi^{19/9}} \\
& + \frac{537568\gamma e_0^2}{5117\chi^{19/9}} - \frac{354035\pi^2\eta e_0^2}{5848\chi^{19/9}} + \frac{2470829204695\eta e_0^2}{1485485568\chi^{19/9}} + \frac{50240\pi^2 e_0^2}{731\chi^{19/9}} - \frac{2058896840770247e_0^2}{2287647774720\chi^{19/9}} - \frac{127825\eta^3}{1296} + \frac{76055\eta^2}{1728} \\
& - \frac{6848\gamma}{21} + \frac{2255\pi^2\eta}{12} - \frac{15737765635\eta}{3048192} - \frac{6848}{21}\log(4x) - \frac{640\pi^2}{3} + \frac{11583231236531}{4694215680}, \\
\alpha_7 = & \frac{13762251650419025\pi e_0^8\eta^2}{14971511771136\chi^{19/9}} - \frac{863950161122145266182945\pi e_0^8\eta^2}{167825277172212645888\chi^{38/9}} + \frac{1919251873874486617026907175\pi e_0^8\eta^2}{245360555225774888288256\chi^{19/3}} \\
& - \frac{40866306288716811342278348158188713\pi e_0^8\eta^2}{11492758825254645564814849769472\chi^{76/9}} - \frac{70352065412362175\pi e_0^8\eta}{29943023542272\chi^{19/9}} + \frac{4416477753219647720024815\pi e_0^8\eta}{335650554344425291776\chi^{38/9}} \\
& - \frac{9811136782221639568423793225\pi e_0^8\eta}{490721110451549776576512\chi^{19/3}} + \frac{208906879936169138297285285341303271\pi e_0^8\eta}{22985517650509291129629699538944\chi^{76/9}} - \frac{14329446184895255375\pi e_0^8}{5030427955101696\chi^{19/9}} \\
& + \frac{899556820693249630490850175\pi e_0^8}{56389293129863449018368\chi^{38/9}} - \frac{1998351515473594240053655597625\pi e_0^8}{82441146555860362464854016\chi^{19/3}} \\
& + \frac{42550561609719173085177224221604751095\pi e_0^8}{3861566965285560909777789522542592\chi^{76/9}} + \frac{19643860461025\pi e_0^6\eta^2}{38304306432\chi^{19/9}} - \frac{116890683997652041915\pi e_0^6\eta^2}{92009472133888512\chi^{38/9}} \\
& + \frac{577566016814470844726725\pi e_0^6\eta^2}{807107089558470027264\chi^{19/3}} - \frac{100418608176175\pi e_0^6\eta}{76608612864\chi^{19/9}} + \frac{597540377518688130805\pi e_0^6\eta}{18401894426777024\chi^{38/9}} \\
& - \frac{2952493765339042903528075\pi e_0^6\eta}{1614214179116940054528\chi^{19/3}} - \frac{20453458379485375\pi e_0^6}{12870246961152\chi^{19/9}} + \frac{121708191973727871410725\pi e_0^6}{30915182636986540032\chi^{38/9}} \\
& - \frac{601369700714292579011030875\pi e_0^6}{271187982091645929160704\chi^{19/3}} + \frac{38630090995\pi e_0^4\eta^2}{168001344\chi^{19/9}} - \frac{35176251579191105\pi e_0^4\eta^2}{201775158188352\chi^{38/9}} - \frac{197475439165\pi e_0^4\eta}{336002688\chi^{19/9}} \\
& + \frac{179819553872611535\pi e_0^4\eta}{403550316376704\chi^{38/9}} - \frac{40222183410925\pi e_0^4}{56448451584\chi^{19/9}} + \frac{36625998186496500575\pi e_0^4}{67796453151286272\chi^{38/9}} + \frac{11625065\pi e_0^2\eta^2}{184212\chi^{19/9}} \\
& - \frac{59426855\pi e_0^2\eta}{368424\chi^{19/9}} - \frac{12104177975\pi e_0^2}{61895232\chi^{19/9}} - \frac{74045\pi\eta^2}{756} + \frac{378515\pi\eta}{1512} + \frac{77096675\pi}{254016},
\end{aligned} \tag{A8}$$

and recall $\chi = f/f_0$ with f_0 the GW frequency of the $\ell = 2$ harmonic at which the eccentricity equals e_0 .

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| <p>[1] O. R. Pols, K.-P. Schröder, J. R. Hurley, C. A. Tout, and P. P. Eggleton, <i>MNRAS</i> 298, 525 (1998).</p> <p>[2] G. M. Harry and LIGO Scientific Collaboration, <i>Class. Quantum Grav.</i> 27, 084006 (2010).</p> <p>[3] F. Acernese <i>et al.</i>, <i>Class. Quantum Grav.</i> 25, 184001 (2008).</p> <p>[4] K. Somiya and the KAGRA Collaboration, <i>Class. Quantum Grav.</i> 29, 124007 (2012).</p> | <p>[5] A. Freise, S. Chelkowski, S. Hild, W. Del Pozzo, A. Pereca, and A. Vecchio, <i>Class. Quantum Grav.</i> 26, 085012 (2009), arXiv:0804.1036.</p> <p>[6] P. C. Peters, <i>Phys. Rev.</i> 136, B1224 (1964).</p> <p>[7] F. Antonini and H. B. Perets, <i>Astrophys. J.</i> 757, 27 (2012).</p> <p>[8] J. Samsing, M. MacLeod, and E. Ramirez-Ruiz, <i>Astrophys. J.</i> 784, 71 (2014), arXiv:1308.2964 [astro-ph.HE].</p> |
|---|--|

- [9] T. A. Thompson, *Astrophys. J.* **741**, 82 (2011), arXiv:1011.4322 [astro-ph.HE].
- [10] W. E. East, S. T. McWilliams, J. Levin, and F. Pretorius, *Phys. Rev. D* **87**, 043004 (2013), arXiv:1212.0837 [gr-qc].
- [11] R. M. O’Leary, B. Kocsis, and A. Loeb, *MNRAS* **395**, 2127 (2009), arXiv:0807.2638.
- [12] B. Kocsis and J. Levin, *Phys. Rev. D* **85**, 123005 (2012), arXiv:1109.4170 [astro-ph.CO].
- [13] D. Tsang, *Astrophys. J.* **777**, 103 (2013), arXiv:1307.3554 [astro-ph.HE].
- [14] L. Wen, *Astrophys. J.* **598**, 419 (2003), arXiv:astro-ph/0211492.
- [15] W. E. East, F. Pretorius, and B. C. Stephens, *Phys. Rev. D* **85**, 124009 (2012), arXiv:1111.3055 [astro-ph.HE].
- [16] D. Pooley, W. H. G. Lewin, S. F. Anderson, H. Baumgardt, A. V. Filippenko, B. M. Gaensler, L. Homer, P. Hut, V. M. Kaspi, J. Makino, B. Margon, S. McMillan, S. Portegies Zwart, M. van der Klis, and F. Verbunt, *Astrophys. J. Lett* **591**, L131 (2003), astro-ph/0305003.
- [17] M. Freitag, P. Amaro-Seoane, and V. Kalogera, *Astrophys. J.* **649**, 91 (2006), astro-ph/0603280.
- [18] F. Antonini and D. Merritt, *Astrophys. J.* **745**, 83 (2012), arXiv:1108.1163 [astro-ph.GA].
- [19] F. Antonini, ArXiv e-prints (2014), arXiv:1402.4865 [astro-ph.GA].
- [20] C. Hopman and T. Alexander, *Astrophys. J. Lett* **645**, L133 (2006), astro-ph/0603324.
- [21] Y. Kozai, *Astronomical Journal* **67**, 591 (1962).
- [22] M. L. Lidov, *Planetary and Space Science* **9**, 719 (1962).
- [23] F. Antonini, N. Murray, and S. Mikkola, *Astrophys. J.* **781**, 45 (2014), arXiv:1308.3674 [astro-ph.HE].
- [24] S. J. Aarseth, *MNRAS* **422**, 841 (2012), arXiv:1202.4688 [astro-ph.SR].
- [25] M. C. Miller and D. P. Hamilton, *Astrophys. J.* **576**, 894 (2002), arXiv:astro-ph/0202298.
- [26] T. J. Maccarone, A. Kundu, S. E. Zepf, and K. L. Rhode, *Nature (London)* **445**, 183 (2007), astro-ph/0701310.
- [27] J. A. Irwin, T. G. Brink, J. N. Bregman, and T. P. Roberts, *Astrophys. J. Lett* **712**, L1 (2010), arXiv:0908.1115 [astro-ph.HE].
- [28] J. Strader, L. Chomiuk, T. J. Maccarone, J. C. A. Miller-Jones, and A. C. Seth, *Nature (London)* **490**, 71 (2012), arXiv:1210.0901 [astro-ph.HE].
- [29] A. C. Sippel and J. R. Hurley, *MNRAS* **430**, L30 (2013), arXiv:1211.6608 [astro-ph.GA].
- [30] M. Morscher, S. Umbreit, W. M. Farr, and F. A. Rasio, *Astrophys. J. Lett* **763**, L15 (2013), arXiv:1211.3372 [astro-ph.GA].
- [31] W. H. Lee, E. Ramirez-Ruiz, and G. van de Ven, *Astrophys. J.* **720**, 953 (2010).
- [32] B. W. Murphy, H. N. Cohn, and P. M. Lugger, *Astrophys. J.* **732**, 67 (2011), arXiv:1205.1049 [astro-ph.GA].
- [33] E. Pfahl, S. Rappaport, and P. Podsiadlowski, *Astrophys. J.* **573**, 283 (2002), astro-ph/0106141.
- [34] R. M. O’Leary, F. A. Rasio, J. M. Fregeau, N. Ivanova, and R. O’Shaughnessy, *Astrophys. J.* **637**, 937 (2006), arXiv:astro-ph/0508224.
- [35] S. R. Kulkarni, P. Hut, and S. McMillan, *Nature (London)* **364**, 421 (1993).
- [36] S. Sigurdsson and L. Hernquist, *Nature (London)* **364**, 423 (1993).
- [37] S. F. Portegies Zwart and S. L. W. McMillan, *Astrophys. J. Lett* **528**, L17 (2000), astro-ph/9910061.
- [38] K. Martel and E. Poisson, *Phys. Rev. D* **60**, 124008 (1999), gr-qc/9907006.
- [39] T. Cokelaer and D. Pathak, *Classical and Quantum Gravity* **26**, 045013 (2009), arXiv:0903.4791 [gr-qc].
- [40] D. A. Brown and P. J. Zimmerman, *Phys. Rev. D* **81**, 024007 (2010).
- [41] E. A. Huerta and D. A. Brown, *Phys. Rev. D* **87**, 127501 (2013), arXiv:1301.1895 [gr-qc].
- [42] S. Babak, H. Fang, J. R. Gair, K. Glampedakis, and S. A. Hughes, *Phys. Rev. D* **75**, 024005 (2007), arXiv:gr-qc/0607007.
- [43] J. R. Gair and K. Glampedakis, *Phys. Rev. D* **73**, 064037 (2006), arXiv:gr-qc/0510129.
- [44] K. Glampedakis and D. Kennefick, *Phys. Rev. D* **66**, 044002 (2002).
- [45] P. A. Sundararajan, G. Khanna, S. A. Hughes, and S. Drasco, *Phys. Rev. D* **78**, 024022 (2008), arXiv:0803.0317 [gr-qc].
- [46] K. Ganz, W. Hikida, H. Nakano, N. Sago, and T. Tanaka, *Progress of Theoretical Physics* **117**, 1041 (2007), gr-qc/0702054.
- [47] Y. Mino, M. Sasaki, and T. Tanaka, *Progress of Theoretical Physics Supplement* **128**, 373 (1997), arXiv:gr-qc/9712056.
- [48] Y. Mino, M. Sasaki, and T. Tanaka, *Phys. Rev. D* **55**, 3457 (1997), arXiv:gr-qc/9606018.
- [49] Y. Mino, M. Sasaki, M. Shibata, H. Tagoshi, and T. Tanaka, *Progress of Theoretical Physics Supplement* **128**, 1 (1997), arXiv:gr-qc/9712057.
- [50] E. Berti, *Classical and Quantum Gravity* **23**, 785 (2006), astro-ph/0602470.
- [51] M. Campanelli, C. O. Lousto, H. Nakano, and Y. Zlochower, *Phys. Rev. D* **79**, 084010 (2009), arXiv:0808.0713 [gr-qc].
- [52] L. Barack and C. Cutler, *Phys. Rev. D* **69**, 082005 (2004), arXiv:gr-qc/0310125.
- [53] N. J. Cornish and J. S. Key, *Phys. Rev. D* **82**, 044028 (2010), arXiv:1004.5322 [gr-qc].
- [54] J. S. Key and N. J. Cornish, *Phys. Rev. D* **83**, 083001 (2011), arXiv:1006.3759 [gr-qc].
- [55] J. Levin, S. T. McWilliams, and H. Contreras, *Classical and Quantum Gravity* **28**, 175001 (2011), arXiv:1009.2533 [gr-qc].
- [56] B. Mikóczy, B. Kocsis, P. Forgács, and M. Vasúth, *Phys. Rev. D* **86**, 104027 (2012), arXiv:1206.5786 [gr-qc].
- [57] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, *Phys. Rev. D* **77**, 064034 (2008), arXiv:0711.0250 [gr-qc].
- [58] R. Gold and B. Brügmann, *Phys. Rev. D* **88**, 064051 (2013), arXiv:1209.4085 [gr-qc].
- [59] I. Hinder, F. Herrmann, P. Laguna, and D. Shoemaker, *Phys. Rev. D* **82**, 024033 (2010).
- [60] T. Hinderer, A. Buonanno, A. H. Mroué, D. A. Hemberger, G. Lovelace, H. P. Pfeiffer, L. E. Kidder, M. A. Scheel, B. Szilagyi, N. W. Taylor, and S. A. Teukolsky, *Phys. Rev. D* **88**, 084005 (2013), arXiv:1309.0544 [gr-qc].
- [61] N. Yunes, K. G. Arun, E. Berti, and C. M. Will, *Phys. Rev. D* **80**, 084001 (2009), arXiv:0906.0313 [gr-qc].
- [62] L. Blanchet, *Living Reviews in Relativity* **9**, 4 (2006).
- [63] M. Boyle, D. A. Brown, L. E. Kidder, A. H. Mroué,

- H. P. Pfeiffer, M. A. Scheel, G. B. Cook, and S. A. Teukolsky, *Phys. Rev. D* **76**, 124038 (2007).
- [64] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: asymptotic methods and perturbation theory*. (Springer-Verlag, 1999).
- [65] M. Tessmer and G. Schaefer, *Phys. Rev. D* **D82**, 124064 (2010), arXiv:1006.3714 [gr-qc].
- [66] M. Tessmer and G. Schaefer, *Annalen Phys.* **523**, 813 (2011), arXiv:1012.3894 [gr-qc].
- [67] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, *Phys. Rev. D* **77**, 064035 (2008), arXiv:0711.0302 [gr-qc].
- [68] D. A. Brown, P. Kumar, and A. H. Nitz, *Phys. Rev. D* **87**, 082004 (2013), arXiv:1211.6184 [gr-qc].
- [69] A. Buonanno, B. R. Iyer, E. Ochsner, Y. Pan, and B. S. Sathyaprakash, *Phys. Rev. D* **80**, 084043 (2009).
- [70] C. Van Den Broeck and A. S. Sengupta, *Classical and Quantum Gravity* **24**, 155 (2007), gr-qc/0607092.
- [71] K. G. Arun, B. R. Iyer, B. S. Sathyaprakash, and S. Sinha, *Phys. Rev. D* **75**, 124002 (2007), arXiv:0704.1086 [gr-qc].
- [72] M. Trias and A. M. Sintes, *Phys. Rev. D* **77**, 024030 (2008), arXiv:0707.4434 [gr-qc].
- [73] A. M. Sintes and A. Vecchio, *ArXiv General Relativity and Quantum Cosmology e-prints* (2000), gr-qc/0005058.
- [74] A. M. Sintes and A. Vecchio, in *American Institute of Physics Conference Series*, American Institute of Physics Conference Series, Vol. 523, edited by S. Meshkov (2000) pp. 403–404, gr-qc/0005059.
- [75] T. A. Moore and R. W. Hellings, *Phys. Rev. D* **65**, 062001 (2002), gr-qc/9910116.
- [76] R. W. Hellings and T. A. Moore, *Classical and Quantum Gravity* **20**, 181 (2003), gr-qc/0207102.
- [77] C. Van Den Broeck and A. S. Sengupta, *Classical and Quantum Gravity* **24**, 1089 (2007), gr-qc/0610126.
- [78] K. G. Arun, B. R. Iyer, B. S. Sathyaprakash, S. Sinha, and C. van den Broeck, *Phys. Rev. D* **76**, 104016 (2007), arXiv:0707.3920.
- [79] E. K. Porter and N. J. Cornish, *Phys. Rev. D* **78**, 064005 (2008), arXiv:0804.0332 [gr-qc].
- [80] K. G. Arun, A. Buonanno, G. Faye, and E. Ochsner, *Phys. Rev. D* **79**, 104023 (2009), arXiv:0810.5336 [gr-qc].
- [81] B. Paczynski, *Astrophys. J. Lett* **308**, L43 (1986).
- [82] D. Eichler, M. Livio, T. Piran, and D. N. Schramm, *Nature (London)* **340**, 126 (1989).
- [83] S. Barthelmy *et al.*, *Nature (London)* **438**, 994 (2005).
- [84] J. Grindlay, S. Portegies Zwart, and S. McMillan, *Nature Physics* **2**, 116 (2006).
- [85] E. Troja, S. Rosswog, and N. Gehrels, *Astrophys. J.* **723**, 1711 (2010), arXiv:1009.1385 [astro-ph.HE].
- [86] R. M. O’Leary, R. O’Shaughnessy, and F. A. Rasio, *Phys. Rev. D* **76**, 061504 (2007), arXiv:astro-ph/0701887.
- [87] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (San Francisco: W.H. Freeman and Co., 1973, 1973).
- [88] C. Königsdörffer and A. Gopakumar, *Phys. Rev. D* **71**, 024039 (2005), gr-qc/0501011.
- [89] C. M. Will and A. G. Wiseman, *Phys. Rev. D* **54**, 4813 (1996), arXiv:gr-qc/9608012.
- [90] C. Cutler, D. Kennefick, and E. Poisson, *Phys. Rev. D* **50**, 3816 (1994).
- [91] L. Blanchet and G. Schaefer, *MNRAS* **239**, 845 (1989).
- [92] A. Gopakumar and B. R. Iyer, *Phys. Rev. D* **65**, 084011 (2002), gr-qc/0110100.
- [93] C. Königsdörffer and A. Gopakumar, *Phys. Rev. D* **73**, 124012 (2006), gr-qc/0603056.
- [94] N. Yunes and E. Berti, *Phys. Rev. D* **D77**, 124006 (2008), arXiv:0803.1853 [gr-qc].
- [95] Z. Zhang, N. Yunes, and E. Berti, *Phys. Rev. D* **D84**, 024029 (2011), arXiv:1103.6041 [gr-qc].
- [96] M. Favata, *Physical Review Letters* **112**, 101101 (2014), arXiv:1310.8288 [gr-qc].
- [97] C. Capano, Y. Pan, and A. Buonanno, *ArXiv e-prints* (2013), arXiv:1311.1286 [gr-qc].
- [98] L. Pekowsky, J. Healy, D. Shoemaker, and P. Laguna, *Phys. Rev. D* **87**, 084008 (2013), arXiv:1210.1891 [gr-qc].
- [99] E. Berger, *ArXiv e-prints* (2013), arXiv:1311.2603 [astro-ph.HE].
- [100] D. Clausen, S. Sigurdsson, and D. F. Chernoff, *MNRAS* **428**, 3618 (2013), arXiv:1210.8153 [astro-ph.HE].
- [101] B. F. Schutz, *Classical and Quantum Gravity* **28**, 125023 (2011), arXiv:1102.5421 [astro-ph.IM].
- [102] M. Dominik, E. Berti, R. O’Shaughnessy, I. Mandel, K. Belczynski, C. Fryer, D. Holz, T. Bulik, and F. Panarale, *ArXiv e-prints* (2014), arXiv:1405.7016 [astro-ph.HE].
- [103] K. S. Tai, S. T. McWilliams, and F. Pretorius, *ArXiv e-prints* (2014), arXiv:1403.7754 [gr-qc].
- [104] M. A. McLaughlin, *Classical and Quantum Gravity* **30**, 224008 (2013), arXiv:1310.0758 [astro-ph.IM].
- [105] A. Sesana, *Classical and Quantum Gravity* **30**, 224014 (2013), arXiv:1307.2600 [astro-ph.CO].
- [106] P. A. Seoane, S. Aoudia, H. Audley, G. Auger, S. Babak, J. Baker, *et al.*, *ArXiv e-prints* (2013), arXiv:1305.5720 [astro-ph.CO].
- [107] J. R. Gair, M. Vallisneri, S. L. Larson, and J. G. Baker, *Living Reviews in Relativity* **16**, 7 (2013), arXiv:1212.5575 [gr-qc].
- [108] The LIGO Scientific Collaboration, the Virgo Collaboration, J. Aasi, B. P. Abbott, R. Abbott, T. Abbott, M. R. Abernathy, F. Acernese, K. Ackley, C. Adams, and *et al.*, *ArXiv e-prints* (2014), arXiv:1403.6639 [astro-ph.HE].
- [109] The LIGO Scientific Collaboration, the Virgo Collaboration, J. Aasi, B. P. Abbott, R. Abbott, T. Abbott, M. R. Abernathy, F. Acernese, K. Ackley, C. Adams, and *et al.*, *ArXiv e-prints* (2014), arXiv:1405.1053 [astro-ph.HE].
- [110] E. A. Huerta, P. Kumar, J. R. Gair, and S. T. McWilliams, *ArXiv e-prints* (2014), arXiv:1403.0561 [gr-qc].
- [111] E. A. Huerta, P. Kumar, and D. A. Brown, *Phys. Rev. D* **86**, 024024 (2012), arXiv:1205.5562 [gr-qc].
- [112] E. A. Huerta and J. R. Gair, *Phys. Rev. D* **83**, 044020 (2011), arXiv:1009.1985 [gr-qc].
- [113] E. A. Huerta and J. R. Gair, *Phys. Rev. D* **83**, 044021 (2011), arXiv:1011.0421 [gr-qc].
- [114] E. A. Huerta and J. R. Gair, *Phys. Rev. D* **84**, 064023 (2011), arXiv:1105.3567 [gr-qc].